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LETTER TO THE EDITOR

Fredholm approximants for the Lippmann–Schwinger equation

M L Sharma and D Bondyopadhyay
Department of Physics, University of Delhi, Delhi 7, India

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Abstract. A method based on Fredholm approximants is proposed for the approximate solution of the Lippmann–Schwinger equation. All diagonal Fredholm approximants constructed in this way are shown to be unitary. The method can be generalized to include Bethe–Salpeter and N/D equations.

Exact solution of the Lippmann–Schwinger (LS) equation by matrix inversion is laborious and time consuming and therefore certain approximation schemes are called for in view of the wide applicability of this equation. Balázs (1968) has solved the LS equation when applied to π–π scattering using the Pagels (1965) method but the technique runs into extrapolation difficulties when the infinite integral converges and the method then becomes somewhat inaccurate. In this letter we indicate a method, based on Fredholm approximants introduced by Moiseiwitsch and O'Brien (1970) and Moiseiwitsch (1970), for solving the LS equation approximately. It turns out that the method is easy to apply, has a wide range of applicability and the approximate solutions in certain cases reproduce the unitarity properties of the scattering amplitude.

The Noyes (1965) method is used to regularize the kernel and the partial wave LS equation then takes the following form

$$t_l(k) = \lambda V_l(k)\left(1+4\pi\lambda\int_0^\infty \frac{p'^2\mathrm{d}p'\,V_l(k,p')f(p')}{p'^2-k^2}\right)^{-1} \tag{1}$$

and

$$f(p) = \frac{V_l(p,k)}{V_l(k)} - 4\pi\lambda\int_0^\infty \frac{p'^2\mathrm{d}p'}{p'^2-k^2}\left(V_l(p,p') - \frac{V_l(p,k)\,V_l(k,p')}{V_l(k)}\right)f(p'). \tag{2}$$

For the convenience of solution we shall rewrite these equations in the following compact form

$$t_l(k) = \lambda V_l(k)\left(1+4\pi\lambda\int_0^\infty \mathrm{d}p'\,V_l(k,p')G(k,p')f(p')\right)^{-1} \tag{3}$$

$$f(p) = F(p,k)+\lambda\int_0^\infty \mathrm{d}p'\,K(k,p,p')G(k,p')f(p'). \tag{4}$$

An approximate solution of (4) can be obtained by replacing the integral by a finite sum and then converting it into a set of simultaneous equations. The solutions are then

$$f(p_j) = \sum_{i=1}^{N} \frac{F(p_i,k)D^N(k,p_i,p_j)}{D_\lambda{}^N(k)}. \tag{5}$$

In this expression $D_\lambda^N(k)$ and $D^N(k, p_i, p_j)$ are respectively the Fredholm determinant and the first Fredholm minor defined in the usual notation (Smithies 1958, Whittaker and Watson 1965).

The integral in (3) may also be replaced by a finite sum and substitution of the solution (5) in it yields

$$t_l(k) = \lambda V_l(k)\left(1+4\pi\lambda\delta\sum_{ij}\frac{V(k,p_j)G(k,p_j)F(p_i,k)D^N(k,p_i,p_j)}{D_\lambda^N(k)}\right)^{-1}. \tag{6}$$

The sum

$$\delta\sum_{j=1}^{N} V(k,p_j)G(k,p_j)D^N(k,p_i,p_j)$$

stands for a determinant obtained by replacing the ith row in $D_\lambda^N(k)$ by

$$V_l(k,p_1)G(k,p_1)\delta, \quad V_l(k,p_2)G(k,p_2)\delta, \quad \ldots .$$

If we denote this determinant by $\delta D^N(k,p_i)$ and take the limit $N \to \infty$, then the expression for the scattering amplitude takes the form

$$t_l(k) = \lambda V_l(k)\left(1+4\pi\lambda\int_0^\infty \frac{D(k,p)}{D_\lambda(k)}F(p,k)\,\mathrm{d}p\right)^{-1} \tag{7}$$

where

$$D(k,p) = V_l(k,p)G(k,p) - \lambda\int \mathrm{d}p'\begin{vmatrix} V_l(k,p)G(k,p) & V_l(k,p')G(k,p') \\ K(k,p',p)G(k,p) & K(k,p',p')G(k,p')\end{vmatrix}$$
$$+\frac{\lambda^2}{2!}\int \mathrm{d}p'\int \mathrm{d}p''$$
$$\times\begin{vmatrix} V_l(k,p)G(k,p) & V_l(k,p')G(k,p') & V_l(k,p'')G(k,p'') \\ K(k,p',p)G(k,p) & K(k,p',p')G(k,p') & K(k,p',p'')G(k,p'') \\ K(k,p'',p)G(k,p) & K(k,p'',p')G(k,p') & K(k,p'',p'')G(k,p'')\end{vmatrix} - \ldots . \tag{8}$$

D_λ also has a similar expansion in powers of λ

$$D_\lambda = \sum \lambda^n d_n, \qquad d_0 = 1 \tag{9}$$

where d_n has a well known representation (Smithies 1958) in terms of the traces of the kernel of equation (4)

$$d_n = \frac{(-1)^n}{n!}\begin{vmatrix} \sigma_1 & 1 & 0 & \ldots & 0 & 0 \\ \sigma_2 & \sigma_1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & \sigma_1 & n-1 \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \ldots & \sigma_2 & \sigma_1 \end{vmatrix} \qquad n \geqslant 1 \tag{10}$$

$$\sigma_n = \mathrm{Tr}(KG)^n.$$

Similarly $D(k,p)$ may be written as

$$D(k,p) = \sum_{n=0}^{\infty} \lambda^n Y_n, \qquad Y_0 = V(k,p)G(k,p) \equiv VG. \tag{11}$$

Using (10) it may be shown that Y_n admits the following representation

$$Y_n = \frac{(-1)^n}{n!}\begin{vmatrix} VG & n & 0 & \dots & 0 & 0 \\ VGKG & \sigma_1 & n-1 & \dots & 0 & 0 \\ VG(KG)^2 & \sigma_2 & \sigma_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & \sigma_1 & 1 \\ VG(KG)^n & \sigma_n & \sigma_{n-1} & \dots & \sigma_2 & \sigma_1 \end{vmatrix} \qquad n \geqslant 1. \tag{12}$$

Using (9), (10), (11) and (12) the amplitude $t_l(k)$ in (7) may finally be expressed as

$$t_l(k) = \lambda V_l(k)$$

$$\times\left(1+\frac{\begin{bmatrix}4\pi\lambda\{VGF-\lambda(\sigma_1 VGF-VGKGF) \\ +\frac{1}{2}\lambda^2[VGF(\sigma_1{}^2-\sigma_2)+\sigma_1 VGKGF-VG(KG)^2F]-\dots\}\end{bmatrix}}{1-\lambda\sigma_1+\frac{1}{2}\lambda^2(\sigma_1{}^2-\sigma_2)-(\lambda^3/3!)[\sigma_1(\sigma_1{}^2-\sigma_2)+\sigma_2\sigma_1-\sigma_3]+\dots}\right)^{-1} \tag{13}$$

where

$$VGF \equiv \int \mathrm{d}p'\, V(k,p')G(k,p')F(p',k)$$

$$VGKGF \equiv \int \mathrm{d}p' \int \mathrm{d}p\, V(k,p')G(k,p')K(k,p',p)G(k,p)F(p,k),$$

$t_l(k)$ is then represented as the ratio of two infinite power series in λ; both the numerator as well as denominator series are entire functions of λ.

Fredholm approximants for the amplitude $t_l(k)$ are constructed by terminating the series of $D_\lambda(k)$ and $D(k,p)$. The $[m, n]$ approximant is obtained by retaining terms up to order λ^m in the numerator and λ^n in the denominator in the expression in (13). As examples, [1, 1] and [2, 2] diagonal approximants are given by the following expressions

$$\underset{[1,1]}{t_l(k)} = \frac{\lambda V(k)}{1-\lambda(\sigma_1-4\pi VGF)} \tag{14}$$

$$\underset{[2,2]}{t_l(k)} = \frac{\lambda V(k)(1-\lambda\sigma_1)}{1-\lambda(\sigma_1-4\pi VGF)+\lambda^2[\frac{1}{2}(\sigma_1{}^2-\sigma_2)-4\pi(VGF\sigma_1-VGKGF)]}. \tag{15}$$

From (12) the imaginary part of Y_n may be calculated

$$\operatorname{Im} Y_n = \tfrac{1}{2}\pi k\, V(k) d_n. \tag{16}$$

Using (7), (13) and (16) yields the relation

$$\operatorname{Im} \underset{[n,n]}{(t_l(k))^{-1}} = 2\pi^2 k \tag{17}$$

showing that the diagonal Fredholm approximants constructed in this way are all unitary.

Moiseiwitsch (1970) has used Fredholm approximants for solving the Schrödinger equation for the exponential and square well potentials. These problems are exactly solvable. The results of his calculation with [2, 2] and [3, 3] diagonal approximants generally agree very well with the exact phase shifts. Unitarity preserving properties of the diagonal approximants of (13) are of great importance for application to scattering problems. This method can be easily extended to the Bethe-Salpeter and N/D equations. A weaker but similar approximation scheme has been used earlier by us (Sharma and Bondyopadhyay 1972) for solving N/D equations with inelastic unitarity in π–π scattering. In practice the approximants can be calculated to sufficiently large orders by utilizing well known recurrence relations satisfied by d_n and Y_n.

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